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The Gap Between the Demand for Understanding Introductory Economics Textbooks and the Supply of High School Mathematics– Use Graph Representations as Examples

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This study employed content analysis on one popular introductory economics textbook to explore whether the graph representations in the textbook match those taught in school mathematics. We analyzed the external and internal representations of coordinate graphs in one of the most math-related chapters. The results showed that neither the various external displays nor the internal imagistic kinesthetic representations used in the economics textbook were sufficiently provided in school mathematics. In economics, the ways of labeling variables, segments, lines, and axes are relatively free and do not match those used in mathematics. Static display representations are often used to express the dynamic/action of objects in graphs, which are usually not emphasized in school mathematics. Students have to construct imagistic kinesthetic representation internally when reading graphs in economics textbooks and this experience is not usually provided in school math classes. This study raised an alert to mathematics educators that the traditional way of using and teaching mathematics representations might be reconsidered in the era of interdisciplinarity.

Keyword: internal representation, textbook analysis, imagistic kinesthetic representation, economics readiness, graph representation

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高中數學與大學經濟學課本之間的供需落差－以圖形表徵為例

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本研究探討大學一年級經濟學課本中的圖形表徵是否與學校數學學過的圖形表徵一致。研究採內容分析法，分析課本中與數學最相關的一章中坐標圖形的外在表徵與內在表徵。研究結果發現，經濟學中常使用的各樣外在表徵與內在想像動態表徵在學校數學中並未充分涵蓋。經濟學在變數、線段、線及坐標軸的標記方式較為隨意，與數學用法不盡相同。經濟學中會用靜態的表徵來表示圖形中動態的物件，這是數學中不強調的；在經濟學，學生需要建構內在想像動態表徵來理解圖形，但學校數學並未提供這樣的學習經驗，此研究結果給數學教育者一個警示：數學教育應重新思考數學表徵的使用與教學以符應當今重視跨學科學習的時代。

關鍵字：內在表徵、教科書分析、動態表徵、經濟學準備度、圖形表徵

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I. Introduction

"$Q^D = f(P;\ other\ factors)$or $Q_x = f(P_x,\ related\ goods,\ income,\ prefernce)$" is defined as a demand function, accompanied by its curve with P on the vertical axis and Q on the horizontal axis in most introductory economics textbooks (Chang et al., 2000; Eastin & Arbogast, 2020). Is D in the expression of an exponent? Is $f(P; other\ factors)$ something like a function $f(x)$ learned in high school? Should we consider "P" as an independent variable? Why is the independent variable on the vertical axis? Students may ask these questions when they first see this function and its graph because these differ from the conventions taught in high school mathematics.

Economics is one of the basic college courses using mathematics. The majority of students who majored in social science will take introductory economics in their first year in college. Students' performance in mathematics was found to have a positive impact on students' performance in economics (Mumuni et al., 2010). Many studies pointed out that students' algebra and graphing abilities affected their performance in college-level economics courses (Crowley & Wilton, 1974; Espey, 1997).

It is very often that new concepts are introduced with different kinds of mathematics representations, such as math expressions, tables, and graphs, in an economics textbook. It has been examined that the use of graphs in economics classes would benefit students when learning the principles of economics (Cohn et al.,2001). Therefore, understanding the related math contents, the mathematical graph and symbol representations is the keystone to learning economics. However, studies showed that students faced difficulties in economics graphs. Veloo and Md-Ali (2015) showed that pre-university students faced greater difficulty with graph items compared to economic problem-solving items in an economic achievement test. Khoo (2008, as cited in Veloo & Md-Ali, 2015, p. 264) pointed out that pre-university economics students frequently faced difficulties in understanding the described graphs.

Moreover, freshman-year math focuses on calculus and/or linear algebra, which are not directly related to graph representation in basic economics courses. Are the mathematical graph representations learned in high school enough for college readiness, such as in economics? There are few, if any, papers illustrating the mathematical representations involved in learning economics, so this study would like to initiate this kind of research to explore the discrepancy between the graph representations and their corresponding functions used in mathematics and those used in economics. The relationships of linear functions and the interpretations of their translations are massively discussed in the introductory economics textbooks, so the linear functions will be the focus of this study.

The main research questions are: What are the major differences between the coordinate graph representations of linear functions in the introductory economics textbooks and the representations introduced in high school mathematics classes?

II. Literature review

1. College readiness and mathematical representations

One of the goals of high school mathematics education is to prepare students for college and career readiness. A study (Siri et al., 2016) showed that mathematics was one of the most problematic subjects relating to the transition from high school to university. Researchers have proven that mathematics played a crucial role in learning economics (Velupillai, 2005). This study would like to investigate the gap between high school mathematics and college economics.

Kao (2009) examined the gaps between an introductory economics textbook and high school mathematics textbooks. Most of the math-related contents, including the first derivative and the instantaneous rate of change, were covered in high school mathematics, except the topic of "total derivatives". This gap is not taught in high school and will not be learned in the first year of college calculus until the very last chapters. Besides the analysis of the learning sequence, researchers would like to investigate the gap from the viewpoints of mathematical representations.

Economists explain the concept with words, equations, tables, charts, and mostly graphs. Stern et al. (2003) found that active graphical representation was a powerful tool to transfer knowledge from one economic content area to another. Those learners with high mathematical competencies benefited more from actively constructing graph representations.

Studies (Veloo & Md-Ali, 2015; Zetland et al., 2010) pointed out the difficulties of learning graphs in economics. Johari et al. (2018) found that students had problems visualizing complex graphs and interpreting data in the graph when learning economics. Cohn et al. (2004) examined students' attitudes toward graphs and the relationship between attitudes and learning in the one-semester college introductory economics course. They found that students who had problems reading and interpreting graphs in economics scored poorly in economics courses. These graphs were highly related to linear functions learned in mathematics. Some mathematics studies showed that students faced difficulties when learning linear functions and interpreting cartesian graphs (Postelnicu, 2011). Nathen et al. (2002) found that students showed a lower level of performance in drawing holistic graphs (line or exponential curves) than instance-based graphs (scatter plots). The possible reasons for economics graph-learning difficulties could be the complexity of economics graphs, the ability to "decode" graphs by connecting an economics graph to a similar prior-learned mathematics graph, the difficulties of learning linear functions, etc.

The visible, concrete graph is one of the external representations. The external representations refer to embodied, observable configurations, such as concrete objects, visible images, equations, verbal forms, manipulatives, and computer operations (Goldin & Kaput, 1996; Hiebert & Carpenter, 1992). Different kinds of external representations were studied with various types of configurations (Heddens, 1984; Lesh et al., 1987; Niss, 2003). In a graph representation, learners' actions such as reading, interpreting the meaning of equations, analyzing the slope and intersections, and drawing inferences from graphs in the real world, are anchored by internal representations. The internal representations are the invisible configurations that are encoded in the human brain, and they dominate the interpretation of external representations (Goldin & Kaput, 1996).

Goldin and Kaput (1996) rejected the view that mathematical meaning was inherent in external representations and instead, they proposed the view that mathematical meaning given to external representations was the product of students' interpretive activity. Therefore, Goldin and Kaput (1996) not only distinguished the internal representations from the external representations but also discussed the interactions of the internal and the external representations in imagistic systems and formal systems. They stated that the construction of formal internal representations mostly depended on the interactions between formal and rule-based systems. The external display representations could be linked via internal representations. This linking could be constructed in the mind of a person, and could also be achieved by interactive media. Besides, internal tactile/kinesthetic representation, one of the imagistic cognitive representational systems, was employed to introduce the imagined physical actions by or on the person.

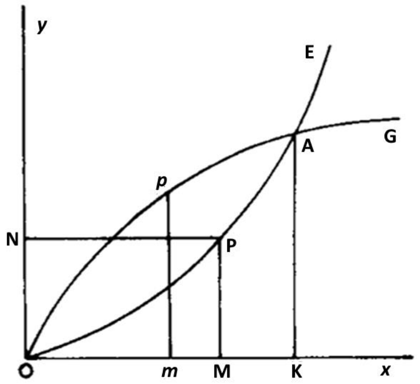
2. Economics graphical representations and elements

The graphical representations in economics were categorized differently in different studies (Chao & Maas, 2017; Demir & Tollison, 2015; Maas & Morgan, 2002). The focused category in this article can be named a causal dependency graph, theory graph, or law curve. These graphs depict the abstract functional relationships between variables, without using real data. This kind of graph exists thoroughly in the economics textbook that we are going to investigate. The famous Marshallian cross diagram is one of the classic economics graphs. The diagram was named after him due to his massive contributions to the design of the diagram (Humphrey, 1996). Therefore, Marshall's diagrams are exemplified for describing the different conventions of graphical elements between mathematics and economics.

Figure 1 shows the foreign trade between England and Germany. The value of all wares which England exported to Germany is measured in yards of cloth, the measurement on the horizontal axis. The value of all wares which Germany exported to England is measured in yards of linen, the measurement on the vertical axis. From mathematics perspectives: E and G refer to curves, x and y refer to the horizontal axis and the vertical axis, and M and N are points on axes. However, in *The pure theory of foreign trade*, curves are named OE and OG. The axes are named as rays, Ox and Oy. The names of points M, N, and P also represent the "length". At point P, the amounts of cloth or linen are represented by OM, ON, and PM. In addition to the curves and points, auxiliary lines are drawn to indicate the coordinates of the points.

Figure 1

Marshall's Figures: Law of International Demand

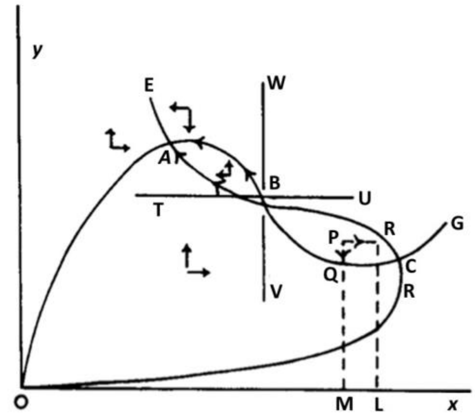


Note. Adopted from "The pure theory of foreign trade," by Marshall, 1879, *History of economic thought books*, p. 29 (https://historyofeconomicthought.mcmaster.ca/marshall/foreigntrade.pdf). In the public domain.

Figure 2 comprises two curves OE, OG, three intersections A, B, C, and many other objects. The graph is not limited to "function", such as OE. This graph explains the status of the equilibrium between curves OE and OG. The unstable equilibrium, B, will move toward the stable position, either A or C. The power that moves B is explained as "forces" that act upon B. Arrowheads are inserted to indicate the directions of the forces which act upon the exchange index at different points. Arrowhead on the curve is to simulate the possible movement from B to A. Besides that, mini coordinate planes are planted in the original diagram. Straight line TBU is drawn from left to right and VBW is drawn vertically upward. This mini coordinate plane is to indicate the location of force by expressing quadrants TBW, VBU, etc.

Figure 2

Marshall's Figures: From Unstable Equilibrium to Stable Equilibrium



Note. Adopted from "The pure theory of foreign trade," by Marshall, 1879, *History of economic thought books*, p. 30 (https://historyofeconomicthought.mcmaster.ca/marshall/foreigntrade.pdf). In the public domain.

Economists had diverse opinions about the location of variables on the causal dependency graph. Take the demand function $Q=f(P)$ as an example. From mathematics perspectives, Q is a function of P, and the independent variable P should be represented on the horizontal axis. Some economists suggested the same way (Cournot, 1838/1897; Dupuit, 1844; Edgeworth, 1889). However, there were other voices. Wicksteed (1888) believed that the variables and function were changeable, in mathematical language, the independent variable and the dependent variable were switchable. Marshall (1879) regarded the opposite way of graphing, and he put Q on the x-axis and P on the y-axis in his diagram. They believed that both ways of graphing had their own interpretations and timing to use.

Dynamic changes are projected onto external representations through mental acts, which means one imagines the motion of the projectile (Goldin & Kaput, 1996). In economics, the models that directly consider the time factor are usually called dynamic (Safiullin & Safiullin, 2018). Schumpeter (1954) defined a relation as dynamic if it connected economic quantities that referred to a different point in time. The dynamic functional relation could be written as $S_t=f(P_{t-1})$ (Ahuja, 2017).

Some "imagined dynamic" motions exist in the graph when defining the stable equilibrium. The equilibrium state of an economy is defined as "static," while the explanation of the movement of industry and trade is defined as "dynamic" (Chao & Maas, 2017). In the graph of stable equilibrium of trade between two countries, Marshall (1879) stated that when the exchange index struck either of the curves in the neighborhood of the equilibrium point, the forces acting on the index tended to "make it oscillate" along the curve toward the equilibrium point. He also analogized the process of "moving" to a mechanics case: "A body may pass through a position of unstable equilibrium on its way toward a position of stable equilibrium". Points are not really oscillating on the graphs, but with the readers' mental act, points are moving. This mental activity is similar to the internal kinesthetic representation proposed by Goldin and Kaput (1996). The dynamic manner in which a path from disequilibrium to equilibrium draws attention in economics classes, while the final static intersection of curves is more concerned in math classes.

III. Research method

Since Goldin and Kaput (1996) and Goldin (1987) cumulated and restructured most, if not all, types of representations used in mathematics and mathematics learning, this study based on the types of representational systems from Goldin and Kaput's studies (Goldin, 1987; Goldin and Kaput, 1996) to analyze the characteristics of graphical representations in economics. In this article, we discussed the discrepancies between the graphical representations in economics and similar graphical representations in mathematics. The researchers chose similar graphical representations in mathematics for comparison because Goldin and Kaput mentioned that when facing representations, we followed those paths of thought that had been previously constructed. The representations in which they were encoded were "available" to us. "Other paths that have never been constructed were not even considered as alternative thoughts, because they are beyond the realm of present imagination.” Therefore, students have nearly no path to follow to understand the economics graphs if they do not refer to similar representations, especially those relating to functions, taught in high school math classes. The relationship between the supply of high school mathematics and the demand for understanding college economics textbooks becomes a critical issue.

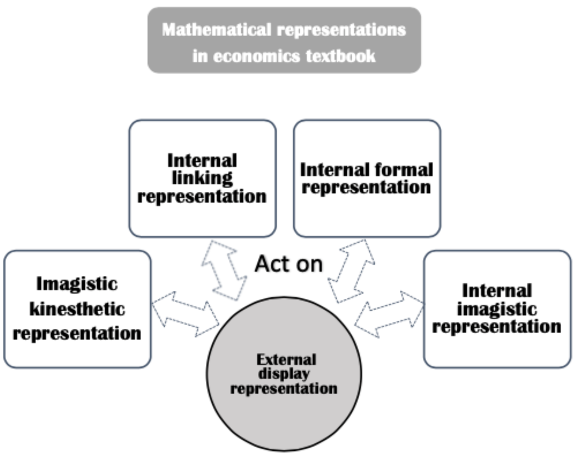
1. Conceptual framework

This research adopted the framework of the system of mathematics representations introduced in Goldin and Kaput’s studies (Goldin, 1987; Goldin and Kaput, 1996). In the focus group discussion (see below in the design and subject session), participants previewed the four economics cases that would be analyzed and excluded the representations which are irrelevant to static textbook analysis from Goldin and Kaput’s studies, such as external action representation, internal auditory/rhythmic representation, internal tactile representation, etc. Then, participants chose the representations that can highlight the differences between mathematics and economics conventions to form this framework, such as external display representations. Participants chose the internal representations to describe the mental configurations which might occur in the process of understanding economics graphs and their functions, such as linking representation, imagistic representation, and internal formal representation. Then, participants renamed some representations based on features, such as imagistic kinesthetic representation focusing on actions, while internal imagistic representation focusing on a static image.

The framework is used to guide the analysis of coordinate graphs in economics textbooks. The framework (Figure 3) included external and internal representations. External representations focus on display representation. Internal representations focus on linking representation, imagistic representation, imagistic kinesthetic representation, and formal representation. Internal representations act on external representations.

Figure 3

The Framework of This Study



Goldin and Kaput (1996) and Goldin (1987) provided the following explanations regarding the system of representations used in our framework. Examples from economics were provided by the authors of this article for clarification. External representations refer to concrete, observable, and physical configurations, such as graphs, equations, tables, manipulatives, etc. The only category we used here is the external display representation. Internal representations refer to possible and not externally observable mental configurations of individuals. Four focuses of our analysis are listed below.

External display representation: the representation used to display relationships between objects, and variables, such as graphs, tables, functions, etc. In the economic context, the demand curve shows the graphical relationship between the quantity demanded and price. The investment function $I = I(i, Y)$ shows that the investment (I) is determined by income (Y) and the marginal propensity to invest (i).

Linking representation: the internal representation that learners can integrate, convert the cognitive structures or mentally link among external representations. External display representations might be linked in the mind of the person who produced them or who reads them (Goldin & Kaput, 1996). Given one external representation, learners interpret and construct the internal configuration or meaning of it. This representation can be used to predict, identify, or produce different external representations of the same entity.

In the economic context, the link between the demand function, demand curve, and demand table can be exemplified. Learners can interpret the graph of the function, and translate the graph into a table. These representations, which show the same quantitative relationship between quantity demanded and price, are being linked by the person who read them. The flexibility of linking and switching among multiple representations benefits learners while solving mathematical problems, and enables learners to understand math notation more effectively (Heinze et al., 2009). More examples of flexibility will be illustrated in the results.

Internal imagistic representation: the representation "included internal imagery, image-schematic representation" (Goldin & Kaput, 1996). They are the internal images when referring to a concept. For example, an imagistic representation of the axiom of equality might be a balance. With the image and mechanism of balance, learners would understand that adding a number on both sides of an equal sign balances the equation. This imagistic representation would help learners understand the axiom of equality easily. Sometimes, this kind of representation conveys meaning in an analogic way. For example, two lines that slant up from left to right are drawn, and the steeper one indicates the larger slope. Two points on the vertical axis are plotted, the high one indicates the larger y-value.

Imagistic kinesthetic representation: When the internal process of imagistic representation is further associated with a kinesthetic component, it is called imagistic kinesthetic representation. It is a dynamic representation, imagined physical actions by or on the person, to construct internal imagistic configurations that appropriately correspond to nonverbal configuration (Goldin & Kaput, 1996). For example, when combining like terms in a quadratic polynomial, students may learn to put all like terms of x together in a parenthesis and all like terms of x^2 together in another parenthesis, as if they were moving the objects into different baskets. These terms are imagined to be moved into baskets like cargo with the learners' imagination.

Sometimes, imagistic kinesthetic representations exist in economics graphs, such as the imagined physical movement of points along the demand curve, indicating the change in demand quantity over price. The imagined physical movements of the whole demand curve indicate the translation of the curves and result in a new equilibrium. The points and the curves become vivid and dynamic with the learners' imagination. More examples will be illustrated in the research results.

Internal formal representation: "the internal representation associated with the construction and manipulations of formal notations" (Goldin & Kaput, 1996). Learners are able to perform the algorithm, discuss and explain symbol manipulation and rules, interpret formal notational descriptions such as symbolic statements and expressions imagistically, and visualize situations formally (Goldin & Kaput, 1996). For example, transform a given algebraic expression into its equivalent form by manipulating math symbols. The saving is the residual income (Y_d), which is not used for consumption (C). The saving function is defined as $S = Y_d - C$. By substituting the consumption function $C = C_a + cY_d$ in the equation, the saving function can be derived into another equation: $S = Y_d - C = Y_d - (C_a + cY_d)$= - $C_a + (1 - c)Y_d$. This revised format in terms of $y = b + mx$ reveals autonomous consumption (C_a) and marginal propensity to save $(1 - c)$.

2. Design and subject

This study was conducted by the researchers in a focus group. The focus group contained six experts in mathematics education. The six experts include a professor, an assistant professor, three Ph.D. students with teaching experiences in high school, and one Ph.D. student with teaching experiences in middle school, ranging from 10 to 30 years. Two economics experts with more than 20 years of teaching experience in high school served as consultants regularly.

The method used to carry out this qualitative study is content analysis. The subject of this study is economics textbooks used in basic economics courses for freshman year. Researchers analyzed the economics textbook "*Economics: Theory and practice*" (Chang et al., 2000). This is one of the most popular textbooks in business schools in Taiwan. The book is written in Chinese, and co-edited by four professors of Economics at National Taiwan University. There are twenty-nine chapters. This study only analyzed Chapter 19, "The simple Keynesian model," in the book. The researchers considered the math relevance in each chapter. It was agreed by the researchers and two consultants who were economic experts that Chapter 19 was one of the chapters that contained the most math functions, equations, and complex graphs and that students required adequate math competency to learn this chapter. This article focuses on a sub-part of the study in which four special cases of graph representations were selected based on criterion sampling (Patton, 2002) and were decided by five 3-hour sections of focus group discussion. The cases met the criterion of appearing to have at least three characteristics different from those in mathematics.

3. Data analysis

This study was conducted in two stages. The first stage was to develop the coding rubric for the external display representation. The characteristics of external display representation of the economics graphs we analyzed include: almost only the first quadrant being illustrated, no arrowheads being drawn on the axes, the independent variable being represented by the vertical axis, schematic diagram being drawn without real values of intercept and slope, multiple auxiliary lines in the graph, the labeling method of an object being various, the value of function sometimes being represented by a line segment. Therefore, the 13-coding rubric for external display representation included quadrant, axis arrowhead, axis label, axis variable, point coordinate, segment label, auxiliary line, characteristic of the line (schematic graph or authentic graph), and representation of the value of the function.

The second stage was to code the chapter's graphs according to the analytical units. Any graph stands alone as an analytical unit, except those that are a series of graphs expressing the same concept with only minor changes. The numbers of codes were summed up for each category of representations. Data triangulations were conducted with three other coworkers and six 3-hr focus group discussion sections, each including at least six experts in mathematics education. Two economics experts served as consultants continuously. Criterion sampling was further used to select special cases for deeper qualitative analysis, which meet the criterion of appearing to have at least three characteristics different from those in mathematics (Patton, 2002). This article reports the analytical results of the four cases selected through criterion sampling.

The conventions of mathematical representations are quite consistently applied in Taiwan's mathematical education, so scholars and experts should be familiar with them. These mathematical conventions are not the focus of this article. However, to provide more rigorous pieces of evidence in comparison, this study analyzed the graph representations in one collection of best-selling high school textbooks for reference. The graphs in this study focus on the relationships between linear functions and their translations, so the researchers analyzed the graphs of polynomial functions taught in high school, which are more relevant to the foundation of understanding linear functions. The analysis results will be described shortly when mentioning comparisons between mathematics and economics representations.

IV. Research result

This article aims to provide initial qualitative results that may show extreme characteristics of graph representations in economics and their relationships with math representations that enable readers to gain a deep understanding and insights. We provide statements regarding the characteristics of the graph representations of four selected cases. These four cases are from three different sections and relate to the determination of the equilibrium point. We illustrate these cases by the external display characteristics which are different from mathematics conventions and two of the internal representations. To introduce these four internal representations thoroughly and fairly, each case is illustrated by two of the internal representations, and each internal representation is reported two times in this article. These four cases will be carried out in a sequence of textbook page numbers.

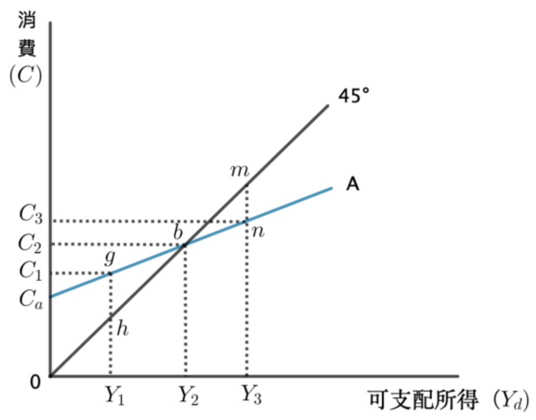
Some economics content is taught in high school Civics and Society. The most related content to this study is "supply and demand." In an unpublished document (Chang, 2023) consisting of economics graphs from all high school Civics and Society textbooks, researchers found that the high school level content only covers the basic concepts and basic graphs, and the graphs in this study are not covered in high school. According to the interview with "some" Civics and Society teachers, they "don't" focus much on graphs while teaching. The graphs in this study are new to learners when they first learn this chapter in college (only translations of curves have been introduced in the previous chapters), therefore they have to apply these representations to read and understand the graphs.

Case 1: Consumption function

The consumption function $C = C_a + cY_d$, line A (see Figure 4), shows the relationship between consumer spending (C) and disposable income (Y_d). C_a is autonomous consumption, a positive value that represents the expenses that consumers must spend even when they have no disposable income. c is the marginal propensity to consume, a value between 0 and 1. The consumption increases while the disposable income increases; the increment of consumption is less than the increment of disposable income. The $45°$ line is drawn for reference, and it shows all points where disposable income and consumption spending are equal. The intersection of line A and the $45°$ line, b, determines the equilibrium.

Figure 4

Consumption Function

Case 1: Consumption function	External display representations which are different from mathematical conventions
	● Arrowhead: No arrowheads at the end of axes ● Line labeling: Line are labeled as C_aA, and 45º line ● Segment labeling: Segment bA, Segment 0bm ● Value of function: When the disposable income is 0, the value of consumption is $0C_a$. When the disposable income is Y_1, the value of consumption is gY_1. ● Line characteristic: (1) The intercept and slope of A are not given numbers. (2) The line is written in form of $y = a + mx$, not $y = mx + a$

Note. The graph is adopted from *Economics: Theory and practice*, by Chang et al., 2000, p.105.

External representation

The display representations between mathematical labeling and economic labeling for the same object have many differences. In mathematics, labeling various constituents of the Cartesian coordinate plane has its rules. Conventionally, lines are labeled by a letter or by expressing two points that the line passes through, with a line notation. Line segments are labeled by expressing the two endpoints of the segment, with a segment notation. For example, $\overline{AB}$ represents the line segment from point A to point B, $\overleftrightarrow{AB}$ represents the line passing through point A and point B, or simply line l. The y-intercept is written as point notation (0, y) or just the value of y.

However, the conventions of labeling segments are relatively free in economics. Segments in the graph are named CaA, bA, and 0bm when mentioned in the textbook. The external display representations are listed in Figure 4. The line segments on the coordinate plane can be labeled with a combination of the point on the coordinate axes and the label of the line ($such\ as\ C_a$A), or a combination of the intersection point and the label of the line ($such\ as$ bA). Occasionally, lines may be labeled by expressing more than two points ($such\ as$ 0bm). The variety in labeling is conjectured that the label of the point has multiple roles. A is not only the label of the line but also the endpoint of the line segment. C_a is not only a value or a point on the y-axis, but also the endpoint of the line segment OC_a. The points on the coordinate plane become part of geometric elements, such as lines or segments. The difference between a line segment and a line seems not to be a big deal in economics. Learners are suggested to have some flexibility towards math-related objects when learning economics.

The displays of the quadrant and the arrowheads on the axes seem different in these two fields. Most of the functional graphs in mathematics textbooks display more than one quadrant and display arrowheads on the axes. Among functional graphs in high school textbooks, 88% of the graphs display four quadrants, and 2% display one quadrant. Regarding the arrowheads on the axes, 91% of the graph displays arrowheads at the ends of the axes. The rest of the graphs, which discuss the concavities and extrema, only show the curve itself and points of inflection, without axes and their arrowheads. However, among the graphs in this economics chapter, 86% of graphs display only one quadrant, and 14% display two quadrants. 100% of the economics graphs don't display arrowheads at the ends of axes.

Linking representation

Without any given numbers for the y-intercept, slope, or any points in the consumption function $C = C_a + cY_d$, the function can still be schematically sketched on the plane. We have the schematic drawing in math, and we usually have $y = mx + b$ as the equation. To extinguish the y-intercept or slope from several unfamiliar parameters, learners can link the external representation of $C = C_a + cY_d$ with the representation of the line equation $y = b + mx$ to help them read the graph and the function. Since the x-axis represents Y_d, and the y-axis represents C, learners can compare the coefficients and realize that the slope is c and the y-intercept is C_a. The different conventions of linear equations in mathematics and economics will be mentioned later.

The 45° line is drawn for reference to compare the disposable income and the consumer spending at a different level of Y_d. Learners must link this representation with the $y = x$ line representation learned in mathematics in order to understand that the points *h*, *b*, and *m* on the 45° line have equal horizontal and vertical distances, respectively.

Internal imagistic representation

When comparing the disposable income and the consumer spending at a different level of Y_d with the 45° line, an imagistic representation of a right isosceles triangle should be activated. When the disposable income is Y_1, the auxiliary line leads to point g, with consumer spending C_1. The right isosceles triangle OhY_1 helps convert the length from OY_1 to hY_1, and convert the point Y_1 to h. Therefore, learners just need to compare h and g, instead of Y_1 and C_1. With the same method, learners just need to compare m and n, instead of Y_3 and C_3. The higher points have a larger value.

Learners must be aware that line A is not a fixed line, but rather is changeable according to different values of slope and intercept. The current line A has a slope of less than 1. Learners must create an image that the consumption line may change to a steeper or smoother line if slope *c* changes. Learners also need to create an image of the effect of y-intercept C_a on line A.

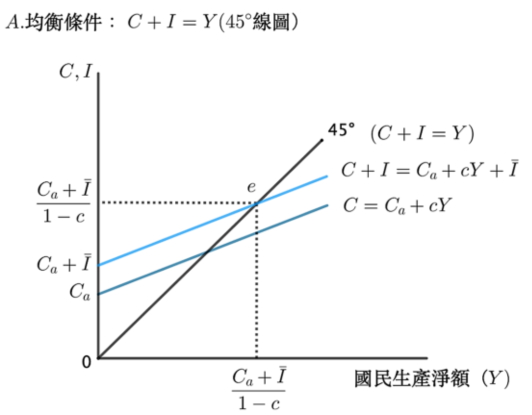
When slope c is defined as a value between 0 and 1, the imagistic representation of line comparison should be activated to understand the reason why *c* should be between 0 and 1. We all know that people who earn more money, even if they spend a lot, still save more money; that is, they have more money in their disposable income than they spend. This situation may be seen from the imagistic representation of the comparison of the steeper 45º line and the less steep consumption function. In mathematics, we prefer to use a more formal way to derive the slope: $\frac{\Delta y}{\Delta x} = \frac{\Delta C}{\Delta Y_d} < 1$, since the increment of consumption is less than the increment of disposable income.

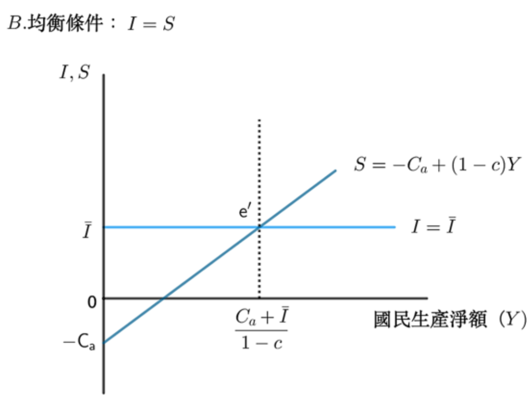
Case 2: Simple Keynes model

This case is to describe the determination of equilibrium under the simple Keynes model (Figure 5). The determination of equilibrium income can be determined by two methods: $C + I = Y$ (graph A) or $I = S$ (graph B).

Figure 5

The Determination of Equilibrium Income

<table>
<tr><td>Case 2: Simple Keynes model</td><td>External display representations which are different from mathematical conventions</td></tr>
<tr><td>




</td><td>
● Arrowhead: No arrowheads at the end of axes
● Axis labeling:
(1) (In A) The vertical axis is labeled as C, I (conjectured: $C + I$)
(2) (In B) The vertical axis is labeled as I, S (conjectured: S or I)
● Line labeling:
(1) 45º line
(2) Lines are labeled by line equation
$C + I = C_a + cY + \bar{I}$
$S = -C_a + (1 - c)Y$
(3) The y-intercept $C_a + \bar{I}$ is separated by the independent variable Y.
● Line characteristic:
(1) The intercepts and slopes are not given numbers.
(2) In graph A, $C = C_a + cY$ is used only as a foundation to graph its translation. $C + I = C_a + cY + \bar{I}$
(3) In graph B, the relationships among three variables are illustrated. The horizontal line $I = \bar{I}$ shows the relationship between I and Y. The slant line $S = -C_a + (1 - c)Y$ shows the relationship between S and Y.
</td></tr>
</table>

Note. The graph A and graph B are adopted from *Economics: Theory and practice*, by Chang et al., 2000, p.118.

In graph A, the horizontal axis is national income, Y. The vertical axis is aggregate demand, $C + I$, the sum of consumption and investment (investment can be viewed as saving or a future benefit that could lead to future consumption), respectively. The graph shows that the intersection of the $45°$ line, $Y = C + I$, and aggregate demand curve $C + I$ is the equilibrium point.

In graph B, the horizontal axis is national income, Y. The national income is determined at the equilibrium point when saving (S) equals investment (I), $I = S$. The investment is a constant number, $I = \bar{I}$. The saving function was derived in the textbook as $S = -C_a + (1 - c)Y$. Graph tells us that the intersection of S and I is the equilibrium point.

Internal formal representation

Internal formal representation is required when reading these graphs. To understand the coordinates of the equilibrium points, learners must be familiar with the variables and their meanings, manipulate the symbols, and work on the algorithm. Three equations under the simple Keynes model are as follows:

(1) $Y = C + I$ The national income (Y) is the sum of consumption (C) and investment (I).

(2) $C = C_a + cY$ The consumption function (C) is determined by national income (Y), while $C_a > 0$ is autonomous consumption, c is the marginal propensity to consume, $1 > c > 0$

(3) $I = \bar{I}$. The investment is a constant number $(\bar{I})$

Graph A indicates that the solution of the system of linear equation $\begin{cases} C + I = Y \\ C + I = C_a + cY + \bar{I} \end{cases}$ is the equilibrium point. Learners have to substitute one equation to the other, obtain the equation $Y = C_a + cY + \bar{I}$, and get the answer $Y = \frac{C_a + \bar{I}}{1 - c}$, which is the x-coordinate and y-coordinate of the equilibrium point. Graph B indicates that the solution of the system of linear equations $\begin{cases} S = -C_a + (1 - c)Y \\ I = \bar{I} \\ I = S \end{cases}$ is the equilibrium point. Learners have to solve the equation to obtain the answer $Y = \frac{C_a + \bar{I}}{1 - c}$, which is also the x-coordinate and y-coordinate of the equilibrium point.

External display representation hinders internal linking representation

In this case, the display representation of labeling and indication of the vertical axis is different from mathematical conventions. The vertical axis is labeled by "C, I" in graph A, and "I, S" in graph B; while the vertical axis is usually labeled by "y", a single independent variable. In economics, due to the complexity of equations, the y-intercepts consist of variables and constants, such as C_a, $C_a + \bar{I}$, $-C_a$. In mathematics, y-intercepts are usually marked with real values, or in form of y_i which represents an unknown value. In the equation $C + I = C_a + cY + \bar{I}$, the y-intercept, $C_a + \bar{I}$ is separated by the independent variable, which barely exists in math classes. These differences hinder learners to apply linking representation to interpret the representation and construct internal configurations which can be used to identify different external representations of the same entity.

The axis is labeled by “C, I” in graph A, and “I, S” in graph B. Two variables are indicated on the vertical axis at the same time, while only one variable, usually y, is indicated in mathematics. Without other descriptions, learners may refer to the union of variables “$C\ or\ I$", the intersection of variables “ $C\ and\ I$ ”, or another condition. These labeling methods hinder learners’ interpretation of the dependent variables. However, the equation $C + I = C_a + cY + \bar{I}$ and its y-intercept $C_a + \bar{I}$ reveal that the vertical axis indicates the value of "$C + I$". If learners notice these representations, then they will be able to link the math ways of representing the y-intercept and y-axis and refer “C, I ” to “$C + I$”. However, this linking process cannot be applied to Graph B, because the only shown y-intercept is $\bar{I}$, a constant value of I, while the dependent variable in the equation $S = -C_a + (1 - c)Y$ is S; these two labels do not match a consistent dependent variable. For graph B, when $Y = 0$, the equation $S = -C_a + (1 - c)Y$ becomes $S = -C_a$. The y-intercept $-C_a$ tells that the vertical axis is S. However, the equation $I = \bar{I}$ also tells that the vertical axis is I. Due to the equivalence of I and S, the labeling “I, S ” should be referred to as “$I\ or\ S$ ”. This is never a representation used in mathematics.

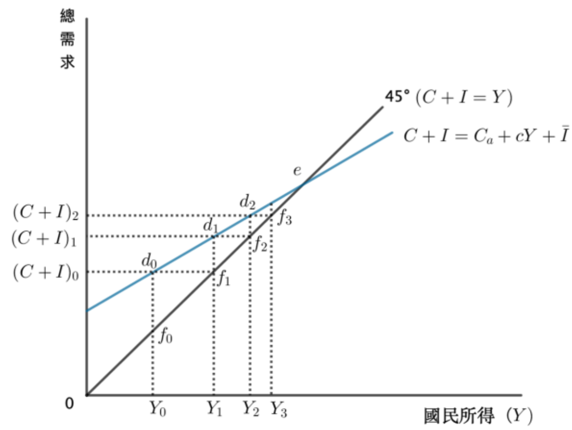
The linking between different conventions might be disturbed by the complexity of curves and massive graphical information loadings. In graph A, the line equation $C = C_a + cY$ is used only as a reference function to graph its translation $C + I = C_a + cY + \bar{I}$. The equilibrium point of graph A has no direct relevance to the line $C = C_a + cY$. Learners must distinguish the useful ones from massive graphical information. In both graphs, the linear equations describe the relationship among three variables: C, I, Y in graph A, and I, S, Y in graph B. In high school math, only two variables are described in the coordinate plane, and three variables are described in space.

Case 3: The process of determination

In Figure 6, the horizontal axis is national income, and the vertical axis is aggregate demand. The $C + I$ curve shows the desired level of demand by consumers. $C + I$ means the sum of consumption and investment, respectively. $(C + I)_i$ on the vertical axis, corresponding to each level of national income, and Y_i on the horizontal axis. The equilibrium is at the point where the level of national income equals aggregate demand.

Figure 6

The Process of the Determination of the Equilibrium Level of Income

Case 3: The process of determination	External display representations which are different from mathematical conventions
	• Arrowhead: No arrowheads at the end of axes • Line labeling: (1) 45º line or $C+I=Y$ (2) Lines are labeled by equation $C+I=C_a+cY+\bar{I}$ • Point labeling: d_i, f_i are labels of the points. They also indicate the "value" of the points $(C+I)_i$ on the vertical axis. • Auxiliary lines: multiple auxiliary lines • Line characteristic: The intercepts and slopes are not given numbers.

Note. The graph is adopted from *Economics: Theory and practice*, by Chang et al., 2000, p.120.

External representation

The labels which are not on the axes, such as d_i, f_i, indicate a point and also a value at the same time (Figure 6). However, it is not easy to tell whether it represents its x-coordinate or its y-coordinate. In this case, it is not obvious whether d_i, f_i correspond to $C+I$ or Y. In addition, the expression $(C+I)_0$ is seldom seen in math classes. The value of the independent variable is usually written explicitly, such as $(f+g)(x_0)$, not $(f+g)_0$. Besides, the expression of $(f+g)(x_0)$ or $(f+g)_0$ is barely adopted on the vertical axis in math classes. It would be difficult for learners to link this kind of representation to something they have learned in math classes.

Imagistic kinesthetic representation

The equilibrium point is the intersection between line $C+I$ curve and 45º line. The point seems static. However, the process of determining the equilibrium level of income is dynamic. The imagistic kinesthetic representation is working in our brain when we read the process.

When the national income is Y_0, the corresponding demand is d_0, which is the value of $(C+I)_0$. The value of Y_0 equals f_0 on the 45º line. We can tell that $d_0 > f_0$, demand is greater than the national income, $(C+I)_0 > Y_0$. The national income is defined as "supply" in the textbook. When the demand d_0 is greater than the supply Y_0, the factory will produce more to meet the demand. Then, the national income will increase to f_1, in order to match the value of d_0 to become a state of balance. But when national income reaches f_1, which is Y_1, the corresponding demand becomes d_1, $(C+I)_0$ changes to $(C+I)_1$. Again, when $d_1 > f_1$, the factory will produce more to meet the supply. The national income will increase to f_2, in order to match the value of d_1 for balance. But when national income reaches Y_2, the corresponding demand becomes d_2; $(C+I)_1$ changes to $(C+I)_2$. After repeating a few cycles, the differences between d_i and f_i are getting smaller, until they are equivalent. At this moment, when the national income equals aggregate demand, there exists the equilibrium point, the intersection of the $C+I$ curve, and the 45^{o} line.

The point is moving along the path: $d_0 \rightarrow f_1 \rightarrow d_1 \rightarrow f_2 \rightarrow \cdots$, until it reaches the equilibrium point e. It seems that learners should wear a pair of special glasses which allow them to see the activities of the points. Taking off the glasses, learners see nothing, but static points lying on the coordinate plane. These physical actions are imagined by the readers, which is the imagistic kinesthetic representation.

Among functional graphs in high school textbooks, 37% of the functional graphs have auxiliary lines. The dashed auxiliary lines show the coordinates of the points or the axis of symmetry of the function. Some auxiliary lines with arrowheads indicate the translation of one function. However, these dashed lines in economic graphs can even show the interaction between two functions and indicate the trace and direction of movements. The existence of lines helps carry out the kinesthetic representation.

Internal imagistic representation

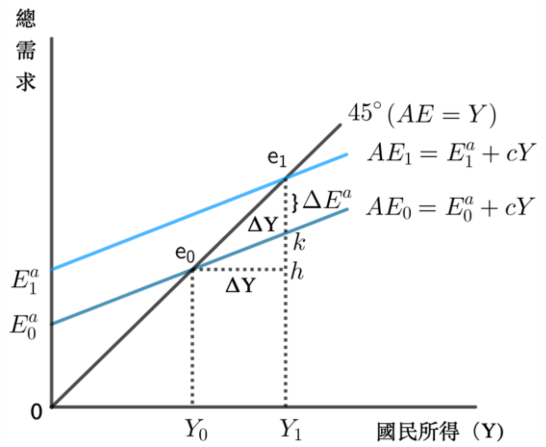
The internal imagistic representation of "limit" is evoked in this case. The movement of the point from $d_0 \rightarrow f_1 \rightarrow d_1 \rightarrow f_2 \rightarrow \cdots$ until the equilibrium point e is similar to the process that $f(x)$ is approaching a specific value when learning the limit in pre-calculus. $\lim_{x \to 1} f(x)$ represents a value that $f(x)$ is approaching when x is getting closer and closer to 1. Mathematics teachers usually move the chalk along the horizontal axis closing to the point $x = 1$ from either the right-hand side or left-hand side, and then move the chalk along the function $f(x)$ approaching $f(1)$. With carefulness, teachers move the chalk infinitely close to $f(1)$, but do not touch it, and draw an arrowhead next to $f(1)$. The movements of points and arrowheads leave an image of "approaching" in learners' minds. This image helps learners understand the trend of the points and their ultimate destination, e. However, a big difference is how to show the path of points in graphs. In mathematics, the point is moving along a smooth curve or line, and the x and y values change at the same time (with y_i's changes depending on x_i's changes); in economics, the point is climbing the staircase formed by auxiliary lines. This kind of "approaching" imagistic representation is rarely seen in high school math classes.

Case 4: Aggregate expenditure function

In Figure 7, the aggregate expenditure function $AE_1 = E_1^a + cY$ shows that the aggregate expenditure is the sum of autonomous expenditure (E^a) and inducted expenditure (cY). Y is the level of national income, and c is the marginal propensity to spend. The inducted expenditure is affected by c, the slope of the aggregate expenditure line. This graph indicates that the change in autonomous expenditure caused the change in the equilibrium point, from e_0 to e_1. An increase in E^a producing a bigger increase in national income is called the multiplier effect.

Figure 7

The Change of the Equilibrium Level of Income

Case 4: Aggregate expenditure function	External display representations which are different from mathematical conventions
	● Arrowhead: No arrowheads at the end of axes ● Line labeling: (1) 45º line is called $0e_0e_1$ (2) Lines are labeled by AE_0, AE_1 ● Value of function: The increment $Y_1 - Y_0$ is referred to as e_0h (a segment) and ΔY(a value) ● Line characteristic: (1) The intercepts and slopes are not given numbers (2) The lines are written in form of $y = a + mx$, not $y = mx + a$ (3) The translation ΔE^a is not reflected in the equation of lines.

Note. The graph is adopted from *Economics: Theory and practice*, by Chang et al., 2000, p.123.

Imagistic kinesthetic representation

The dynamic representation reveals the change of values over a period of time (Ainsworth & VanLabeke, 2004). The change of status is a big issue in economics, and people always want to investigate the change of equilibrium. The increment of autonomous expenditure on the vertical axis causes the translation of AE_0, moving ΔE^a units upward to AE_1, and the movement of equilibrium points from e_0 to e_1. The translation of the line might occur more than one time, and then the equilibrium points will correspondingly move along the 45º line. What we should observe from the graph is like a silhouette, a collection of lines and a collection of points. They are externally static, but their movements should be traced in readers' mind internally.

Some economics graphs show points moving along the curve, one or two curve translations, and accompanying new points of intersection. These lines and points are not really moving but are imagined by the reader. This imagistic kinesthetic representation makes the graphs come alive in our imagination. This imagistic kinesthetic representation might be thought unreal, illusory, or vague; therefore, the authentic dynamic movements should be demonstrated by technology. This representation is projected onto external representations through one's mental act, and many techniques have been developed to assist in the mental act (Goldin & Kaput, 1996). With the operation of interactive media, learners can physically observe the connection among objects and build the corresponding imagistic kinesthetic representation. However, when this kind of representation is shown in economics books, it is externally presented with both the original graph and the moved graph. Learners must construct their internal imagistic kinesthetic process to understand the relationships between the graphs and their corresponding functions.

Internal Formal representation

Compared with the mathematical formal representation of the function and linear expression, the expression $AE_1 = E_1^a + cY$ is hardly connected to the function expression $f(x) = mx + a$ or the linear equation $y = mx + a$ for learners. The key reason is that it is difficult to tell which letter is variable and which letter is constant. Additionally, the order of the items is reversed. The linear equation is usually expressed in the form of $y = mx + a$ in math classes, while $y = a + mx$ is applied in the economics textbook.

In mathematics, the equation of the function and its vertically translated function are usually represented as $y = f(x)$ and $y = f(x) + k$, in which the translated length and direction are emphasized. However, the "characteristics of translation" are not focused in economics, but the "phenomena of parallel lines". In mathematics, the general forms of parallel lines are usually represented as $f(x) = mx + a$ and $f(x) = mx + b$. With the internal formal representation, learners can rewrite the equation, from $AE_1 = E_1^a + cY$ and $AE_0 = E_0^a + cY$ to $AE_1 = cY + E_1^a$ and $AE_0 = cY + E_0^a$, to interpret the slope of parallel lines, the y-intercept, and the change of lines.

There seems to be little or no experience with adopting a superscripted index and a subscripted index at the same time, such as E_1^a and E_0^a, in math classes. The superscripted indexes in mathematics are the powers or exponents, and subscripted indexes are usually for counters. In economics, the superscripted indexes are used for indication, usually written in terms of the abbreviation, such as Q^D for quantity demand and E^a for autonomous expenditure. The subscripted indexes can be used for counters, time periods, or item indications. Economists investigate the phenomenon of shifting lines and their consequences; therefore, multiple indices are required to show their sequences and identities. The combination of superscript and subscript indexes is often applied in economics.

The effect that an increase in E^a produces a bigger increase in national income is called the multiplier effect, and the multiplier is defined as $\frac{the\ change\ of\ national\ income}{the\ change\ of\ autonomous\ expendidure}$. This can be represented with a math symbol, $\frac{\Delta Y}{\Delta E^a}$. The usage of the symbol ΔY and ΔE_a is similar to the case when learning slope, integration rule, the definition of derivative in mathematics, and the distance of a segment. The value of multiplier $\frac{\Delta Y}{\Delta E^a}$ is greater than 1, since $\Delta Y > \Delta E^a$ which can be observed from the graph.

With the internal formal representation, learners can manipulate the equation and rewrite it into an understandable equation. They can also recognize the mathematical symbols in math-related equations, which helps interdisciplinary communication. Besides, learners must be aware of different conventions in different fields. For example, learners have to get used to seeing "Y" on the horizontal axis in economics, especially when they have learned "Y" as a dependent variable on the vertical axis in math classes.

V. Conclusion

According to the research results of four cases, there are major differences between the graph representations in the economics textbook and the representations learned in high school math classes.

The study shows that the external display representations of the coordinate plane in introductory economics are significantly different from those in high school mathematics, such as the complexity of curves, graphical information loading, the indication of axes, object labeling, and the number of auxiliary lines. For the complexity of curves and graphical information loading, multiple curves are sketched to describe the original status and the change of phenomena in economics graphs. Sometimes, multiple auxiliary lines and the 45^{o} line are sketched for reference in the same coordinate planes. To find out the relationship between variables, learners must deal with a heavier information loading in economics than in mathematics. For the indication of axes, the dual variables "C, I" on the vertical axis represent "$C + I$" in one graph, and the dual variables "I, S" on the vertical axis represent "$I\ or\ S$" in another graph. The representations of dual variables on the vertical axis have never been taught in high school mathematics. Learners might be confused with the graphs. For object labeling, the line segments can be labeled not only by expressing two endpoints of the segment but also with more flexibility. A line segment is named C_aA, with the point C_a on the vertical axis and the label of line A. For the auxiliary lines, there are many more auxiliary lines in economics graphs than in mathematics. The auxiliary lines not only indicate the coordinates of the points but also indicate the moving direction of the imagistic movements of the object, especially when discussing the change of equilibrium point.

The conventions of external display representations are quite different in mathematics and economics. The variety and flexibility of economics representations have broadened our views. Mathematical representations tend to express a static moment in scenarios, but situations in the real world are usually dynamic. When these mathematical representations are applied to different scenarios, there must exist some flexibility to adapt to new scenarios. This flexibility can be introduced appropriately in high school math classes. If we can add some cases with flexible usages in different scenarios in math textbooks- for example, how to link the representations in economics back to the mathematical representations, then it would enlarge the practical value of mathematics and increase the economics readiness in the future.

Linking representation is for learners to integrate or mentally link among external representations. There are often differences in the representations of the same objects between economics and mathematics, which reduces the smooth link between representations. For example, to tell that the consumption function $C = C_a + cY_d$ and the saving function $S = -C_a + (1-c)Y$ are lines, students have to link them with the line equation $y = mx + a$ learned in high school mathematics. However, the appearance of these two function expressions is hardly linked to lines.

Imagistic kinesthetic representation makes the graphs come alive in learners' imaginations. In the process of determining equilibrium, the equilibrium point doesn't appear at once, but "moves" along the curve until it reaches the equilibrium point. However, in high school math, equilibrium is only considered the intersection of two curves, and it is static but not dynamic. This representation tends to help learners understand the relationship between the original graph and the moved graph, which is not sufficiently illustrated in high school math. With the assistance of technology devices, this kinesthetic representation corresponding to objects can be better built up.

With internal formal representation, learners can figure out the intersection of curves by manipulating the symbols and working on the algorithm. However, the mathematical process for solving the point of intersection is not a big focus in economics. The relationship between variables and the process of approaching the equilibrium point is much more important in economics.

Internal imagistic representation evokes learners' existing mental images to learn new content. However, the mathematical image is not always the same as the economical one. In the process of determining equilibrium, a change in the y-value or x-value makes the point move toward equilibrium. The path of points on the graph is like a staircase. In mathematics, the situation where both x and y values change at the same time is discussed more often, and the point is moving along a smooth curve or line in these cases. The way to show the path of points in graphs is quite different.

VI. Suggestion

Transfer of learning occurs when prior-learned knowledge and skills affect how knowledge and skills are learned (Cormier & Hagman, 1987). This article is not to discuss which convention should be correct but to point out the gaps that might exist when transferring mathematics background knowledge to economics. Math conventions are taught rigorously and legitimately in math classes. Learners will be confused when they first learn economics. Things are different! Significantly different external display representations may cause difficulty when recognizing math-related objects. The four internal representations need to be activated when applying mathematical background knowledge to economics graphs. Learners need to "transform" the mathematics representations into economics ones, to learn the new context. Learners used to learn one intersection between two lines in the system of linear equations in math classes. Now they have to generalize the concept in economics graphs, to understand the multiple intersections between multiple lines and curves, and the relationships among these intersections.

Some transitions can be illustrated and developed in high school. For example, some cases with flexible external display representation usages in different scenarios can be added to mathematics textbooks or introduced by Civics and Society teachers. With the practice of mathematics software, learners can develop imagistic kinesthetic representation and broaden their internal imagistic representation through hands-on activities or perception. These research results have shown the importance of scaffolding between high school mathematics and introductory economics. The overall goal of instructions is to help learners construct correct internal representations (Cobb et al., 1992). This scaffolding needs to be constructed to provide learners with more readiness, however, by which field or a new interdisciplinary track? It needs further discussion and research.

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